ON MEASURING VIOLATIONS OF THE PROGRESSIVE PRICIPLE AND THE POTENTIAL REDISTRIBUTIVE EFFECT

THE ISSUE OF HORIZONTAL EQUITY AMONG DEMOGRAPHIC AND SOCIAL GROUPS

Measuring *Equity* violations in a tax system

Kakwani and Lambert (1998) specify three axioms, which can be applied to equivalent incomes:

axiom 1 $x_i \geq x_j \Rightarrow t_i \geq t_j$ (minimal progression)

axiom 2 $x_i \geq x_j$ and $t_i \geq t_j \Rightarrow t_i / x_i \geq t_j / x_j$ (progressive principle)

axiom 3 $x_i \geq x_j$, $t_i \geq t_j$ and $t_i / x_i \geq t_j / x_j \Rightarrow x_i - t_i \geq x_j - t_j$.

(the marginal tax rate should not exceed 100%)

KL's notation:

X are pre-tax incomes;

T: taxes;

A: average tax rates ($a_i = t_i / x_i$);

Y: post-tax incomes; $Y = X - T$.

Z may indicate T, A or Y.

How can we check the presence of axiom violations?

Example

ABSOLUTE VALUES RANKED BY *X*

TAX PAYER	X	T	Y	A
A	40	10	30	0.2500000
B	100	40	60	0.4000000
C	125	30	95	0.2400000
D	150	100	50	0.6666667
E	230	110	120	0.4782609
TOTAL	645	290	355	2.0349275

SHARES RANKED BY X

TAX PAYER	X/TOT_X	T/TOT_T	Y/TOT_Y	A/TOT_A
A	0.0620155	0.0344828	0.084507	0.2500000
B	0.1550388	0.137931	0.1690141	0.4000000
C	0.1937984	0.1034483	0.2676056	0.2400000
D	0.2325581	0.3448276	0.1408451	0.6666667
E	0.3565891	0.3793103	0.3380282	0.4782609
TOTAL	1	1	1	2.0349275

LORENZ AND CONCENTRATION COORDINATES RANKED BY X

TAX PAYER	Q_X	Q_T	Q_Y	Q_A
A	0.0620155	0.0344828	0.084507	0.1228545
B	0.2170543	0.1724138	0.2535211	0.3194217
C	0.4108527	0.2758621	0.5211268	0.437362
D	0.6434109	0.6206897	0.6619718	0.764974
E	1	1	1	1

$$G_X = \left[1 - \sum_{i=1}^{5}\left(Q_i^X + Q_{i-1}^X\right)\cdot f_i\right]=0.2667;\ \ C_{T|X} = \left[1 - \sum_{i=1}^{5}\left(Q_i^{T|X} + Q_{i-1}^{T|X}\right)\cdot f_i\right]=0.3586;$$

$$C_{Y|X} = \left[1 - \sum_{i=1}^{5}\left(Q_i^{Y|X} + Q_{i-1}^{Y|X}\right)\cdot f_i\right]=0.1915;\ \ C_{A|X} = \left[1 - \sum_{i=1}^{5}\left(Q_i^{A|X} + Q_{i-1}^{A|X}\right)\cdot f_i\right]=0.1422.$$

REMARK. Here the asymptotic approximation is applied: A_{max}=1/2.

ABSOLUTE VALUES RANKED IN NON DECREASING ORDER

TAX PAYER	X	T	Y	A
A	40	10	30	0.24
B	100	30	50	0.25
C	125	40	60	0.4
D	150	100	95	0.4782609
E	230	110	120	0.6666667
TOTAL	645	290	355	2.0349275

SHARES RANKED IN NON DECREASING ORDER

TAX PAYER	X/TOT_X	T/TOT_T	Y/TOT_Y	A/TOT_A
A	0.0620155	0.0344828	0.084507	0.1179403
B	0.1550388	0.1034483	0.1408451	0.1228545
C	0.1937984	0.137931	0.1690141	0.1965672
D	0.2325581	0.3448276	0.2676056	0.235026
E	0.3565891	0.3793103	0.3380282	0.327612
TOTAL	1	1	1	1

LORENZ COORDINATES (NON decreasing order ranking)

TAX PAYER	Q_X	Q_T	Q_Y	Q_A
A	0.0620155	0.0344828	0.084507	0.1179403
B	0.2170543	0.137931	0.2253521	0.2407948
C	0.4108527	0.2758621	0.3943662	0.437362
D	0.6434109	0.6206897	0.6619718	0.672388
E	1	1	1	1

$$G_X = \left[1 - \sum_{i=1}^{5}\left(Q_i^X + Q_{i-1}^X\right)\cdot f_i\right] = 0.2667;\quad G_T = \left[1 - \sum_{i=1}^{5}\left(Q_i^{T|X} + Q_{i-1}^{T|X}\right)\cdot f_i\right] = 0.3724;$$

;

$$G_Y = \left[1 - \sum_{i=1}^{5}\left(Q_i^{Y|X} + Q_{i-1}^{Y|X}\right)\cdot f_i\right] = 0.2535;\quad G_A = \left[1 - \sum_{i=1}^{5}\left(Q_i^{A|X} + Q_{i-1}^{A|X}\right)\cdot f_i\right] = 0.2126.$$

OBSERVE:

$$(-G_Z \leq C_{Z|X} \leq G_Z).$$

DEFINE:

$$R_{Z|X} = G_Z - C_{Z|X}.$$

$\longrightarrow$ $0 \leq R_{Z|X} \leq 2$

$R_{Z|X}$ is the Atkinson, Plotnick, Kakwani re-ranking index.

$R_{T|X} = G_T - C_{T|X}$=0.3724−0.3586=0.0138;

$R_{Y|X} = G_Y - C_{Y|X}$=0.2535−0.1915=0.0620;

$R_{A|X} = G_A - C_{A|X}$=0.2126−0.0705.

K.L. suggest to check axiom violations by the Atkinson-Plotnick-Kakwani re-ranking index (*APK*):

$$R_{Z|X} = (G_Z - C_{Z|X})$$

G_Z is the concentration Gini index for attribute Z and $C_{Z|X}$ is the concentration index for attribute X, ranked according to the non decreasing order for X.

$$(-G_Z \leq C_{Z|X} \leq G_Z) \longrightarrow 0 \leq R_{Z|X} \leq 2$$

BASIC FORMULAE

Gini index:

$$G_Z = \frac{1}{2\mu_Z N^2} \sum_{i=1}^{K} \sum_{j=1}^{K} \left| z_i - z_j \right| p_i p_j =$$

$$= \frac{1}{2\mu_Z N^2} \sum_{i=1}^{K} \sum_{j=1}^{K} \left(z_i - z_j \right) p_i p_j I_{i-j}^{Z}$$

$$I_{i-j}^{Z} = \begin{cases} 1: & z_i \geq z_j \\ -1: & z_i < z_j \end{cases} \qquad \sum_{i=1}^{K} p_i = N$$

Concentration index:

$$C_{Z|X} = \frac{1}{2\mu_Z N^2} \sum_{i=1}^{K} \sum_{j=1}^{K} \left(z_i - z_j\right) p_i p_j I_{i-j}^{Z|X}$$

$$I_{i-j}^{Z|X} = \begin{cases} 1: & x_i > x_j \\ -1: & x_i < x_j \\ I_{i-j}^{Z}: & x_i = x_j \end{cases}$$

From which is immediate to verify:

$$-G_Z \le C_{Z|X} \le G_Z$$

Re-ranking idex:

$$R_{Z|X} = \frac{1}{2\mu_Z N^2} \sum_{i=1}^{K} \sum_{j=1}^{K} \left(z_i - z_j \right) p_i p_j \left(I_{i-j}^{Z} - I_{i-j}^{Z|X} \right)$$

From which:

$$0 \le R_{Z|X} \le 2 \cdot G_Z$$

KL start from the Kakwani progressivity index:

$$P = C_{T|X} - G_X,$$

(based on the Jakobsson-Fellman and the Jakobsson- Kakwani theorems: Lambert 2001, pp.190-191, 199-200).

If the derivative of the tax rate is non negative, i.e. $a'(x) \geq 0$, then

$$P \geq 0.$$

Consider the tax elasticity $LP(x) = \left[t'(x) / a(x) \right]$.

Given two tax systems, if $LP_1(x) \leq LP_2(x)$

$$\rightarrow \quad P_1 \leq P_2$$

As

$$G_Y = C_{Y|X} + R_{Y|X}$$

$$\tau P = \tau\left[C_{T|X} - G_X\right] = G_X - C_{Y|X} \qquad \tau = \mu_T / \mu_Y ,$$

the redistributive effect (i.e. $RE = G_X - G_Y$) can be written as

$$RE = G_X - G_Y = G_X - C_{Y|X} - R_{Y|X}$$

$$= \tau\left[C_{T|X} - G_X\right] - R_{Y|X} = \tau P - R_{Y|X}$$

Or, as $C_{T|X} = G_T - R_{T|X}$,

$$RE = \tau\left[G_T - R_{T|X} - G_X\right] - R_{Y|X} = \tau\left[G_T - G_X\right] - \tau R_{T|X} - R_{Y|X}$$

$$=\tau\left[P+R_{T|X}\right]-\tau R_{T|X}-R_{Y|X}.$$

Observe $R_{T|X}\geq 0 \longrightarrow P+R_{T|X}=G_T-G_X\geq C_{T|X}-G_X$

KL call

$$\tau\left[G_T-R_{T|X}-G_X\right]=\tau\left[G_T-G_X\right]=\tau\left[P+R_{T|X}\right]$$

the *potential redistributive* effect: it would occur if no tax re-ranking, e.g. if no Axiom 1 violation occurred.

Observe: $R_{Y|X}>0$ is caused by an excess of progressivity.

Example (University of Turin, Public Finance exam, May 2015)

Tax payer	Gross income	tax
A	7	1
B	9	4
C	12	8

-Does the tax respect Axiom 1 and 2?

-Calculate the Kakwani progressivity index $P = C_{T|X} - G_X$;

-Calculate the re-ranking index R^{APK};

-Without modifying the Gini coefficient for the net income distribution, rearrange the tax distribution in order to make R^{APK}=0;

-Check if the new tax distribution respects Axiom 1 and 2, and calculate the new Kakwani progressivity index $P = C_{T|X} - G_X$.

Keeping in mind $\tau\left[G_T - R_{T|X} - G_X\right]$, by analogy, KL define

$$\tau\left[P + R_{T|X} + \left(R_{A|X} - R_{T|X}\right)\right],$$

As the potential redistributive effect that would occur if neither tax re-ranking, nor tax-rate re-ranking occurred, e.g. if neither Axiom 1 nor Axiom 2 violation occurred.

If we define

$S_1 = \tau R_{T|X}$ is the loss due to violations of Axiom 1

$S_2 = \tau\left(R_{A|X} - R_{T|X}\right)$ is the loss due to violations of Axiom 2

$S_3 = R_{Y|X}$ is the loss due to violations of Axiom 1, Axiom 2 and Axiom 3,

The redistributive effect can be written as

$$RE = \tau\left[P + R_{T|X} + \left(R_{A|X} - R_{T|X}\right)\right] - S_1 - S_2 - S_3$$

Consequently K.L. measure the extent of each axiom violations by the statistics:

Axiom 1 (minimal progression):

$$\tau R_{T|X} = \tau\left(G_T - C_{T|X}\right) = S_1 \qquad\qquad \tau = \frac{\Sigma t_i}{\Sigma y_i}$$

Axiom 2 (progressive principle):

$$\tau\left(R_{A|X} - R_{T|X}\right) = \tau\left[\left(G_A - C_{A|X}\right) - \left(G_T - C_{T|X}\right)\right] = S_2$$

Axiom 3 (the marginal tax rate should not exceed 100%):

$$R_{Y|X} = \left(G_Y - C_{Y|X}\right)$$

$\tau R_{T|X} = \tau\left(G_T - C_{T|X}\right)$ is zero if no violation occurs for Axiom 1, conversely it is positive when Axiom 1 is violated somewhere.

Analogous considerations apply to

$\tau\left(R_{A|X} - R_{T|X}\right) = \tau\left[\left(G_A - C_{A|X}\right) - \left(G_T - C_{T|X}\right)\right]$: Axiom 2;

$R_{Y|X} = \left(G_Y - C_{Y|X}\right)$: Axiom 3

Observe:

$\tau R_{T|X} \geq 0,\ \tau R_{Y|X} \geq 0$ necessarily;

$\tau\left(R_{A|X} - R_{T|X}\right) \geq 0$ practically always

(KL 1998, Mazurek & Vernizzi 2013)

EXAMPLE

$$RE = \tau\left[P + R_{T|X} + \left(R_{A|X} - R_{T|X}\right)\right] - S_1 - S_2 - S_3 =$$

$$0.2667 - 0.2535 =$$

$$= 0.8169\left[\left(0.3586\text{-}0.2667\right) + 0.0138 + \left(0.0705 - 0.0138\right)\right] +$$

$$- 0.8169 \cdot 0.0138 - 0.8169 \cdot 0.0705 - 0.0620.$$

We focus axiom violations occurring between different family typologies, measuring both the extent and the direction of violations.

Gini and concentration indexes can be calculated by different approach: we make use of differences between pairs (pairs of equivalent incomes, taxes, tax rates) associated to indicators functions).

Traditionally researches consider only overall violations

$$R_{T|X},\ R_{A|X} \text{ and } R_{Y|X}.$$

However when dealing with heterogeneous families, **it is very important to detect the intensity and the direction of violations** across families, who have to face different needs.

How much is the tax system aware that a 3.50 zł *kremówki* costs 14 zł when you have wife and 2 children?

THE ITALIAN PERSONAL TAX SYSTEM

Monti, Pellegrino, Vernizzi (2012) estimate axiom violations by making use of the Bank of Italy Survey on Household Income and Wealth, moreover they evaluate the proportion and the direction of axiom violations among 5 family typologies:

s one person family

c couple with one or two incomes

***c*+1** couple with one child (one or more incomes)

***c*+2** couple with two children (one or more incomes)

c+3 couple with three or more children (one or more incomes)

MPV adopt Kakwani and Lambert's equivalence scale

$$sd_h = \left(ad_h + 0.2ch_{1,h} + 0.4ch_{2,h} + 0.7ch_{3,h}\right)^{0.8} + 0.1w_h$$

ad = number of adults

ch_1 = number of children aged 5 years or less

ch_2 = number of children aged between 6 and 14 years

ch_3 = number of children aged between 15 and 17 years

w = number of employees or self-employed within the families

Axiom 1: shares and directions of violation extents between family typologies

$\frac{R_T^{h>j}}{R_T^{h,j}}\%$		j				
		s	c	$c1$	$c2$	$c3$
h	s	-	34.82	14.34	10.83	9.70
	c	65.18	-	25.23	17.69	15.76
	$c1$	85.66	74.77	-	38.46	34.55
	$c2$	89.17	82.31	61.54	-	47.46
	$c3$	90.30	84.24	65.45	52.54	-

Axiom 2: shares and directions of violation extents between family typologies

$\frac{R_A^{h>j}}{R_A^{h,j}}\%$		j				
		s	c	$c1$	$c2$	$c3$
h	s	-	40.51	20.79	17.75	22.69
	c	59.49	-	25.65	19.26	18.59
	$c1$	79.21	74.35	-	40.73	41.39
	$c2$	82.25	80.74	59.27	-	52.31
	$c3$	77.31	81.41	58.61	47.69	-

Axiom 3: shares and directions of violation extents between family typologies

$\frac{R_Y^{h>j}}{R_Y^{h,j}}\%$		j				
		s	c	$c1$	$c2$	$c3$
h	s	-	67.31	85.83	91.30	93.40
	c	32.69	-	74.31	83.14	85.52
	$c1$	14.17	25.69	-	59.08	68.67
	$c2$	8.70	16.86	40.92	-	58.21
	$c3$	6.60	14.48	31.33	41.79	-